

Exact Phase Space Localized Projectors from Energy Eigenstates

J.J.Halliwell

Blackett Laboratory

Imperial College

London SW7 2BZ

UK

Abstract

In investigations of the emergence of classicality from quantum theory, a useful step is the construction of quantum operators corresponding to the classical notion that the system resides in a region of phase space. The simplest such constructions, using coherent states, yield operators which are *approximate* projection operators – their eigenvalues are approximately equal to 1 or 0. Such projections may be shown to have close to classical behaviour under time evolution and these results have been used to prove some useful results about emergent classicality in the decoherent histories approach to quantum theory. Here, we show how to use the eigenstates of a suitably chosen Hamiltonian to construct *exact* projection operators which are localized on regions of phase. We elucidate the properties of such operators and explore their time evolution. For the special case of the harmonic oscillator, the time evolution is particularly simple, and we find sets of phase space localized histories which are exactly decoherent for any initial state and have probability 1 for classical evolution. These results show how approximate decoherence of histories and classical predictability for phase space histories may be made exact in certain cases.

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I. INTRODUCTION

The question of the how classical mechanics emerges from quantum mechanics continues to attract much interest [1]. There are many different aspects to this question, but this paper concerns one specific aspect, namely, the question of characterizing phase space localization in quantum mechanics and the time evolution of phase space localized states.

In classical mechanics, we may construct a phase space probability distribution function $w(p, q)$ that is perfectly localized in a phase space cell Γ . The time evolution of such a probability distribution is straightforward – it evolves to another distribution perfectly localized about a cell Γ_t , the evolution of the original cell along the classical orbits. (See Fig.(1)). In fact, the existence and evolution properties of such distribution functions characterize classical behaviour. In looking for the emergence of classical behaviour from quantum theory a reasonable approach is to attempt to reproduce these steps in the quantum theory as closely as possible.

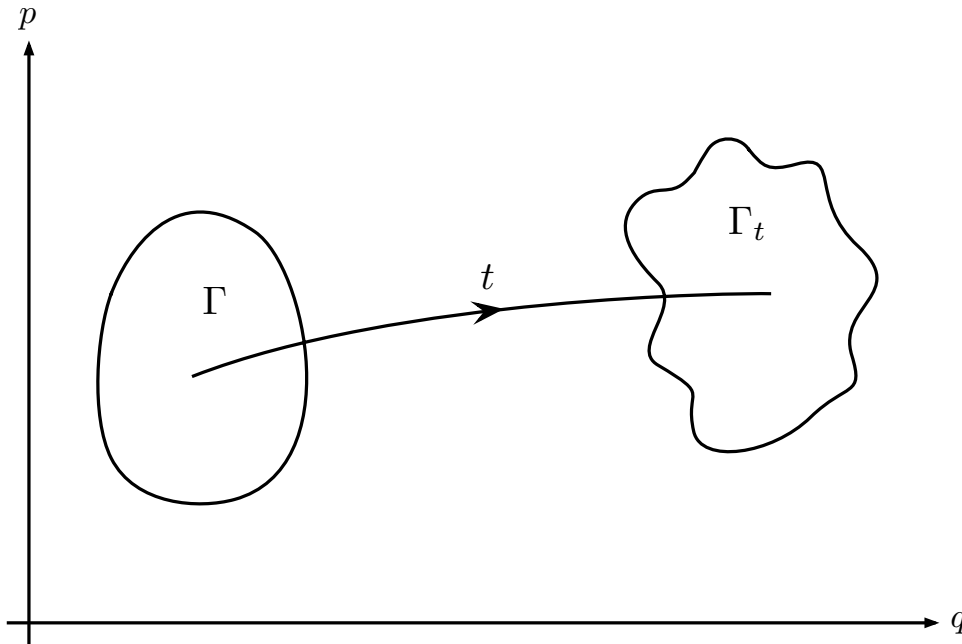


FIG. 1: The classical evolution of a phase space cell Γ .

Of course in quantum theory we do not expect to be able to reproduce these properties exactly. First of all, the fact that positions and momenta are represented by non-commuting operators means that it will not be possible to find states that are perfectly localized in phase

space. Secondly, except for linear systems, the evolution of quantum systems in phase space (in the Wigner function picture, for example [2–4]), follows the classical orbits at best only approximately.

Omnès has made some interesting progress in this area [5, 6]. He characterizes a phase space state localized in a region Γ of phase space by a quasi-projector of the form

$$P_\Gamma = \int_\Gamma dp dq |\psi_{pq}\rangle \langle \psi_{pq}| \quad (1.1)$$

where $|\psi_{pq}\rangle$ are a set of Gaussian states localized in p and q (with the coherent states as a special case). These operators are not exact projectors, however, since the defining property of a projection operator

$$P^2 = P \quad (1.2)$$

is satisfied only approximately by P_Γ , due to the fact that the states $|\psi_{pq}\rangle$ are only approximately orthogonal. Bounds may be found on the quantity $\|P_\Gamma^2 - P_\Gamma\|$ (in a suitably defined norm) and this bound give the limits to within which P_Γ may for practical purposes be regarded as a projector. The time evolution of these quasi-projectors may be determined using results of Hagedorn on the evolution of Gaussian states in a general potential [7]. This leads to the interesting result

$$e^{\frac{i}{\hbar}Ht} P_\Gamma e^{-\frac{i}{\hbar}Ht} \approx P_{\Gamma(-t)} \quad (1.3)$$

with a calculable error [5, 6], where $\Gamma(-t)$ denotes the classical evolution (backwards in time) of the initial phase space cell Γ .

In brief, these properties of the quasi-projector Eq.(1.1) in the quantum case meet classical expectations up to certain estimable errors. However, this leaves the question as to whether there is room for improvement, in terms of making some of these approximations exact, at least in certain special cases. There are a number of reasons why one might want to do this.

First of all, when one talks about an “approximation”, one usually has in mind an exact result which may be approached in some limit. The above results only become exact in the limit $\hbar \rightarrow 0$, which is not really a physically meaningful limit. It might therefore be of interest to find some non-trivial situations in the quantum theory in which exact projectors exist, with exactly specifiable time evolution properties. Such exact results would then form a useful background from which approximations to the more general case could be constructed.

The second reason concerns the decoherent histories approach to quantum theory, one of the most useful frameworks in which to address emergent classicality [5, 6, 8–12]. Omnès has shown that his results above allow one to define sets of phase space localized histories with the property of approximate decoherence [5, 6]. Exact projectors with exactly defined time evolution would give exact decoherence, thereby giving some insight into the relationship between exact and approximate decoherence of histories.

The purpose of this paper is to show how to construct a class of exact projection operators with phase space localization. For linear systems, their time evolution is exactly classical.

A similar task was attempted in a previous paper [13]. There, inspired by an old idea of von Neumann [14], operators \hat{X}, \hat{P} were sought which are in a certain sense close to the usual position and momentum operators, \hat{x}, \hat{p} , but which commute, $[\hat{X}, \hat{P}] = 0$. From the spectrum of these operators it is possible to construct a set of exact projection operators P_{Γ_i} , with phase space localization, satisfying

$$\sum_i P_{\Gamma_i} = 1, \quad P_{\Gamma_i} P_{\Gamma_j} = \delta_{ij} P_{\Gamma_i} \quad (1.4)$$

where Γ_i denotes a set of rectangular phase space cells covering the entire phase space. (This is in turn related to older work on the orthogonalization of the coherent states [15–17]). Although this work achieved to some extent the goals set out here, it suffers from some weaknesses. The states used in the construction of the projection operators were quite complicated. More significantly, the whole construction only works for rectangular (or piecewise rectangular) cells that are quite large compared to a quantum cell of size \hbar . In addition, the initial density operator of the system must be coarse-grained over cells of about the same size. For example, cells of size $10^6 \hbar$ gave good results, and are comfortably adequate for discussions of the emergent classicality of macroscopic systems, but one wonders whether it would be possible to find a construction with much smaller phase space cells.

Here we achieve precisely that. We find exact projectors requiring considerably less coarse graining of the initial state to give sensible results. This improvement comes with a price, namely that we can construct a set of just two projectors, P_{Γ} and its complement $\bar{P}_{\Gamma} = 1 - P_{\Gamma}$, not the infinite set P_{Γ_i} obtained in Ref.[13]. However, this is in fact sufficient for discussions of emergent classicality. Furthermore, the properties of these projectors are considerably simpler under time evolution. In the remainder of this paper we set $\hbar = 1$ for convenience.

II. EXACT PROJECTORS FOR CIRCULAR PHASE SPACE REGIONS

A. The Strategy

How do we modify the quasi-projector Eq.(1.1) into an exact projection operator? This object is a positive hermitian operator and may therefore be diagonalized,

$$P_\Gamma = \sum_n \lambda_n |\psi_n\rangle \langle \psi_n| \quad (2.1)$$

where the eigenstates $|\psi_n\rangle$ are orthogonal and $\lambda_n \geq 0$. From the properties of Eq.(1.1) one can anticipate that the eigenstates $|\psi_n\rangle$ will be phase space localized states and, more importantly, since P_Γ is almost a projector, most of its eigenvalues will be close to 1 or 0. This suggests the following strategy for constructing an exact projector: replace the eigenvalues λ_n with eigenvalues $\tilde{\lambda}_n$ that are exactly 1 or 0. The resulting object,

$$E_\Gamma = \sum_n \tilde{\lambda}_n |\psi_n\rangle \langle \psi_n| \quad (2.2)$$

will be an exact projector which we expect will have the desired phase space localization properties. (There will clearly be some ambiguity in this procedure for the eigenvalues in the transitional region from 0 to 1 but it seems unlikely this ambiguity will be significant).

B. Circular Phase Space Regions

It is unlikely that this strategy can be carried out in general, for arbitrary regions Γ . However, it can be carried out explicitly for circular regions in phase space with the Gaussian functions $|\psi_{pq}\rangle$ in Eq.(1.1) taken to be the standard coherent states $|z\rangle$. We take these coherent states to be those of the Hamiltonian

$$K = \frac{1}{2}(p^2 + q^2) \quad (2.3)$$

where $z = (q + ip)/\sqrt{2}$ and we consider a quasi-projector

$$P_\Gamma = \int_\Gamma \frac{d^2 z}{\pi} |z\rangle \langle z| \quad (2.4)$$

where Γ is a circular region of radius R centred on the origin. We show that this quasi-projector is in fact diagonal in the eigenstates $|n\rangle$ of the Hamiltonian K . Using $z = re^{i\theta}$ and

using the fact that

$$\langle z|n\rangle = \frac{z^n}{(n!)^{\frac{1}{2}}} e^{-|z|^2/2} \quad (2.5)$$

we have

$$P_\Gamma = \int_0^R r dr \int_0^{2\pi} \frac{d\theta}{\pi} |z\rangle \langle z| \quad (2.6)$$

$$= \sum_{n,m} \int_0^R r dr \int_0^{2\pi} \frac{d\theta}{\pi} \frac{(z^*)^n z^m}{(n!)^{\frac{1}{2}} (m!)^{\frac{1}{2}}} e^{-|z|^2} |n\rangle \langle m| \quad (2.7)$$

The crucial result is that the θ integral produces a Kronecker delta,

$$\int_0^{2\pi} \frac{d\theta}{\pi} e^{i(m-n)\theta} = 2\delta_{nm} \quad (2.8)$$

so we have

$$P_\Gamma = \sum_n \lambda_n |n\rangle \langle n| \quad (2.9)$$

where

$$\lambda_n = \frac{2}{n!} \int_0^R dr r^{2n+1} e^{-r^2} \quad (2.10)$$

The dominant contribution to the r integral comes from $r = n^{\frac{1}{2}}$ and it is easily seen that $\lambda_n \approx 0$ for $R \ll n^{\frac{1}{2}}$ and $\lambda_n \approx 1$ for $R \gg n^{\frac{1}{2}}$. The oscillator eigenstates $|n\rangle$ are not phase space localized in the same sense as coherent states, for example, but it is well-known that the eigenstate $|n\rangle$ is concentrated inside the circle of radius $n^{\frac{1}{2}}$, that is, inside the classical orbit corresponding to that value of energy. (This will be discussed more below).

The underlying reason that the quasi-projector is diagonal in oscillator eigenstates in this case is easy to see. Eq.(2.4) clearly satisfies

$$e^{iKs} P_\Gamma e^{-iKs} = P_\Gamma \quad (2.11)$$

since K simply shifts the coherent states along the classical orbits, which are circles in this case, and this maps the domain of integration Γ into itself. So K commutes with P_Γ and they may be diagonalized in the same basis.

We thus see that the diagonalization of the quasi-projector is carried out very easily in this case. Following the general strategy, a simple modification of the quasi-projector leads to the exact projector

$$E_\Gamma = \sum_{n=0}^N |n\rangle \langle n| \quad (2.12)$$

We anticipate that this object is localized about a circular region in phase space of radius $N^{\frac{1}{2}}$ centred about the origin, so N needs to be chosen so that $N^{\frac{1}{2}} = R$ for Eq.(2.12) to approximate Eq.(2.4).

C. Phase Space Localization Properties

We now demonstrate the localization properties of Eq.(2.12) in more detail. Consider the probability associated with the phase space region Γ , as defined using the exact projector Eq.(2.12). It may be written in terms of the Wigner functions of E_Γ and the state ρ as

$$\begin{aligned} p(\Gamma) &= \text{Tr}(E_\Gamma \rho) \\ &= 2\pi\hbar \int dp dq W_E(p, q) W_\rho(p, q) \end{aligned} \quad (2.13)$$

The Wigner function of E_Γ is easily obtained from the Wigner functions of the harmonic oscillator states. It is,

$$W_E(p, q) = \sum_{n=0}^N \frac{(-1)^n}{\pi} L_n(r^2) e^{-r^2/2} \quad (2.14)$$

where $r^2 = 2(p^2 + q^2)$ and L_n are the Laguerre polynomials [2]. This will be non-zero and rapidly oscillating for small r . For large r the $L_n(r^2)$ goes like r^{2n} and each term in the summand decays exponentially for $r \gg n^{\frac{1}{2}}$. It follows that $W_E(p, q)$ will be small for $r \gg (2N)^{\frac{1}{2}}$.

By contrast, the Omnès quasi-projector, Eq.(1.1), will have a Wigner function that takes an approximately constant positive value for small r , decaying to zero for large r . Hence the price of an exact projector versus a quasi projector is the rapid oscillations, which may mean that the exact projector does not have good localization properties for certain types of states.

It is also enlightening to look at the probability $p(\Gamma)$ using the P -function representation [4] of the density operator,

$$\rho = \int d^2z P(z) |z\rangle\langle z| \quad (2.15)$$

We then have

$$p(\Gamma) = \int d^2z P(z) \langle z | E_\Gamma | z \rangle \quad (2.16)$$

where

$$\begin{aligned}\langle z|E_{\Gamma}|z\rangle &= \sum_{n=0}^N |\langle n|z\rangle|^2 \\ &= \sum_{n=0}^N \frac{|z|^{2n}}{n!} e^{-|z|^2}\end{aligned}\tag{2.17}$$

This is approximately 1 for $|z|^2 \ll N$ and approximately zero for $|z|^2 \gg N$. Hence the probability is

$$p(\Gamma) \approx \int_{|z|^2 < N} d^2z P(z)\tag{2.18}$$

This probability will be physically sensible as long as the P -function is positive. If it is negative, positive and negative parts of P may cancel in the integration yielding an unrepresentatively small value for the probability for certain regions of phase space. It is well-known that coarse graining of the P function over phase space regions of size order \hbar make it positive [18], which suggests that Eq.(2.18) will therefore give good phase space probabilities as long as the region integrated over is larger than a few Planck-sized cells. This therefore represents a big improvement on the earlier work Ref.[13] which required coarse graining over a large number of Planck cells.

D. More General Phase Space Regions

The construction Eq.(2.12) yields an exact projector localized around a circle in phase space centred around the origin. This is simply generalized in a number of ways. To obtain a projector centred about another point p, q in phase space, we apply the unitary shift operator

$$U(p, q) = \exp(ip\hat{q} - iq\hat{p})\tag{2.19}$$

The projector

$$E_{pq} = U(p, q)E_{\Gamma}U^{\dagger}(p, q)\tag{2.20}$$

is then localized to a circle centred around the point (p, q) . One may also apply standard unitary operators to E_{pq} which produce squeezing and rotations in phase space, turning the circular regions into ellipses of any eccentricity and orientation [3, 4]. Hence it is possible to construct an exact projector localized about any elliptical region in phase space.

Exact projectors localized to general elliptical regions could be useful even if the dynamics generates phase space cells of more complicated shapes – one can use the projector to ask if the system is within the smallest elliptical region containing the given phase space cell of complicated shape. This could be useful if the boundary is not too irregular. Still, it is of interest to generalize the construction given so far.

For regions Γ in phase space of more general shape, the diagonalization procedure for the projector Eq.(1.1) is unlikely to be implementable in practice. However, the construction of Eq.(2.12) suggests another strategy which may be useful for some geometries. This is to keep the same form Eq.(2.12) but to *choose* the Hamiltonian operator K used in its construction so that its eigenstates $|n\rangle$ are localized in a given region Γ of any reasonable shape. More precisely, suppose that the region Γ is a generic region centred around the origin with boundary $\partial\Gamma$. It will probably be necessary to require that $\partial\Gamma$ is reasonably smooth. Clearly $\partial\Gamma$ is a closed curve around the origin, $(p(s), q(s))$ where s is periodic. Classically, it seems very plausible that, at least for some interesting class of boundary curves, we can *choose* the potential U in a Hamiltonian

$$K = \frac{p^2}{2} + U(q) \quad (2.21)$$

so that the boundary curve $\partial\Gamma$ is one of its integral curves, with fixed energy K_Γ , say. Furthermore, integral curves with lower values of energy will be closed curves lying inside $\partial\Gamma$.

When we look at the spectrum of \hat{K} in the quantum theory, it will have properties similar to the classical paths. They will also be qualitatively similar to the spectrum of the simple harmonic oscillator case already considered. This can be seen, for example, from studies of the Wigner function of the eigenstates of bound state Hamiltonians [19]. That is, the eigenstates $|n\rangle$ up to some value $N \approx 2K_\Gamma$ will be localized in Γ and decay rapidly outside. Hence, the projector of the form Eq.(2.12) will have the desired property of localization in the region Γ . In essence what we are doing here is distorting the construction for the case where K is a simple harmonic oscillator Hamiltonian and Γ a circular region, in such a way that the circular region is turned into a general region.

This procedure clearly will not work for any region Γ – the main restriction is that the curves must be reflection-symmetric under $p \rightarrow -p$. Furthermore, this construction has not been spelled out in detail, so is only offered as a suggestion. However, it does indicate that a

construction for some more general and interesting geometries is possible.

III. TIME EVOLUTION

We now consider the time evolution of the exact projectors on circular regions defined in the previous section. At this point we stress that there are potentially two Hamiltonian operators to consider. One is the simple harmonic oscillator Hamiltonian K associated with the coherent states defined above. This is simply an operator used in the definition of the projectors. The other is the physical Hamiltonian of the system in question, which we denote H , used to evolve the system in time. These two operators need not be the same. However, we here consider the case in which $H = K$ and consider the time evolution of the projector Eq.(2.20).

We first note that the projector E commutes with H , so the time evolution of E_{pq} is entirely in the unitary shift operators Eq.(2.19). We also note that

$$e^{iHt}U(p, q)e^{-iHt} = \exp(ip\hat{q}(t) - iq\hat{p}(t)) \quad (3.1)$$

where

$$\hat{q}(t) = \hat{q} \cos t + \hat{p} \sin t \quad (3.2)$$

$$\hat{p}(t) = -\hat{q} \sin t + \hat{p} \cos t \quad (3.3)$$

It follows that

$$e^{iHt}U(p, q)e^{-iHt} = U(p(-t), q(-t)) \quad (3.4)$$

where $p(t)$ and $q(t)$ are defined in the same way as Eqs.(3.2), (3.6). Applying these results to Eq.(2.20), we find

$$\begin{aligned} e^{iHt}E_{pq}e^{-iHt} &= U(p(-t), q(-t))E_{pq}U^\dagger(p(-t), q(-t)) \\ &= E_{p(-t)q(-t)} \end{aligned} \quad (3.5)$$

That is, unitary time evolution of a projector centred on the point p, q consists simply of shifting it (backwards in time) along the classical trajectory with initial data p, q .

This is similar to the appealing earlier result Eq.(1.3) except that this result is exact. It is exact firstly, because the system is linear, but secondly, because of the special relationship

between the system Hamiltonian and the construction of the projector, which means that the projector commutes with the Hamiltonian (which is not true of the Omnès quasi-projectors).

A more general situation to consider is that in which the projectors onto circular regions are subject to time evolution under a more general Hamiltonian, so K and H are different. We expect a similar quasiclassical result to hold. However, classically, a circular region of phase space evolving under a general Hamiltonian will not remain circular, so it is necessary to consider the construction of exact projectors onto phase space regions which are not circular.

IV. EXACTLY DECOHERENT PHASE SPACE HISTORIES

We now briefly describe the connection of the above results to exact decoherence of histories. In the decoherent histories approach to quantum theory [5, 6, 8–12], histories are represented by projection operators P_α at each moment of time, where $P_\alpha P_{\alpha'} = \delta_{\alpha\alpha'} P_\alpha$ and $\sum_\alpha P_\alpha = 1$. It is sufficient for what we are doing here to focus on the simplest non-trivial case of histories characterized by two moments of time and in this case the candidate probability for a set of histories is

$$p(\alpha_1, \alpha_2) = \text{Tr} (P_{\alpha_2} e^{-iHt} P_{\alpha_1} \rho P_{\alpha_1} e^{iHt}) \quad (4.1)$$

For these to be satisfactory probabilities, the condition of decoherence must be satisfied, which means that the decoherence functional,

$$D(\alpha_1, \alpha'_1, \alpha_2) = \text{Tr} (P_{\alpha_2} e^{-iHt} P_{\alpha_1} \rho P_{\alpha'_1} e^{iHt}) \quad (4.2)$$

is zero for $\alpha_1 \neq \alpha'_1$. This condition is typically satisfied only approximately (which has led to some discussion as to what this actually means).

There are, however, some simple circumstances in which the decoherence condition is satisfied exactly. The first and obvious one is that in which the projectors commute with H , that is, they are histories of conserved quantities. The second and perhaps more subtle one, is that in which there exists an exact relation of the form,

$$e^{iHt} P_{\alpha_2} e^{-iHt} = P_{\alpha_1(t)} \quad (4.3)$$

where the label $\alpha_1(t)$ depends on α_2 and on t . There is clearly exact decoherence in this case, as long as the labels α_1 and α_2 are properly chosen. In simple terms it arises because the quantum theory has a sort of exact determinism, more general than simple conservation.

The phase space projectors considered here provide an example of this sort of determinism and hence of exact decoherence. To see this, we consider the simple harmonic oscillator example above and choose P_{α_1} and P_{α_2} to be exact projections onto circular phase space regions Γ_1 and Γ_2 related by classical evolution (and their complements). We therefore let α_1 take two values and let the projectors P_{α_1} denote the projector E_{pq} onto a phase space region Γ_1 , defined above in Eq.(2.20), and its negation, $\bar{E}_{pq} = 1 - E_{pq}$. Similarly, we let P_{α_2} denote the projector onto the time-evolved phase space region Γ_2 , $E_{p(t)q(t)}$ and its negation, $1 - E_{p(t)q(t)}$. We immediately see from Eq.(3.6) that these projectors satisfy a relation of the form Eq.(4.3) and there is exact decoherence. To see this explicitly, a typical off-diagonal term in the decoherence function is

$$D(\alpha_1, \alpha'_1, \alpha_2) = \text{Tr} \left(e^{iHt} E_{p(t)q(t)} e^{-iHt} E_{pq} \rho \bar{E}_{pq} \right) \quad (4.4)$$

which is clearly zero (from Eq.(3.6) with t replaced by $-t$) since the projectors are all exact projectors.

This analysis is easily extended to a set of phase space localized projectors at an arbitrary sequence of times. Given exact decoherence, one then can consider the probabilities for such phase space histories. We therefore consider a sequence of phase space regions $\Gamma_1, \Gamma_2, \dots, \Gamma_n$, which are all related by classical evolution. It is the easy to see that the probability for a set of histories characterized by projectors of the above type is quite simply $\text{Tr}(E_{pq}\rho)$, since a relation of the form Eq.(3.6) holds and the projectors are exact, so a string of such projectors collapses down to just one. In fact, we could choose the initial state to be the phase space localized state $\rho = E_{pq}/\text{Tr}(E_{pq})$, in which case the probability of the above sequence of phase space regions is then exactly 1, so there is an exact classical determinism in this model.

These results are an exact version, in the special case of a simple harmonic oscillator, of the approximate decoherence discussed by Omnès [5, 6], using the approximate relation Eq.(1.3).

V. SUMMARY AND CONCLUSIONS

We have shown how to construct exact projection operators which are localized about regions of phase space. The construction was given and explored in detail for circular regions and easily extends to elliptical regions of arbitrary centre, eccentricity and orientation

using unitary transformations. We sketched how the construction might proceed for more general regions. The time evolution of projections onto circular regions was considered. It is particularly simple and was used to give an example of exactly decoherent sets of phase space histories, with an exact classical determinism.

VI. ACKNOWLEDGEMENTS

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